Lower Bounds on Deformations of Dynamically Loaded Rigid-Plastic Continua

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In a series of papers published over the last several years, J. B. Martin has established and demonstrated the use of techniques for obtaining upper bounds on displacements of dynamically loaded continua. This paper presents a technique for obtaining lower bounds on the deformation of dynamically loaded rigid-plastic continua and thus complements the work of Martin. Included in the paper is a derivation of a new lower bound theorem, a description of suitable application techniques and the solution of several example problems involving impulsively loaded beams and plates. For these problems, comparisons are made between the predictions of the lower bound theorem developed here, the exact solution, and the upper bounds determined by Martin. The example problems discussed are of the impulsive type, i.e., an initial velocity distribution is imposed on the system and the surface tractions are assumed to do no work during the response of the continuum.

Nomenclature

A^*	= amplitude of assumed deformation mode
$D(\dot{\epsilon}_{ij}^*)$	= dissipation energy rate function = $\sigma_{ij}^*\dot{\epsilon}_{ij}^*$
F_i	= field of body forces
$J(\sigma_{ij})$	= convex yield function
$\dot{k}_r^*,\dot{k}_\theta^*$	= assumed radial and circumferential curvature rates
L	= length of beam
M	= rigid mass
M_0	= limiting or yield plastic moment
m	= mass per unit length
m_A	= mass per unit area
Q_i^*	= assumed dimensionless time independent mode shape
R	= radius of circular plate
r	= radial dimension
S	= surface area of the continuum
T_i	= set of surface tractions
T^*	= $(t_f^* - t)/t_f^*$ = time dependent deformation function
t	= time variable
t_f	= response time of inelastic system
t_f^*	= response time corresponding to assumed deformation
U_i^*	= assumed time independent deformation mode
u_i	= actual deformation field
u_i^s	= assumed time independent kinematically admissible velocity field
u,g,w	= actual deformation components
U^*,G^*,W^*	= assumed deformation mode components
$\dot{u}_i,\ddot{u}_i$	= actual velocity and acceleration fields
$\dot{u}_i^0$	= initial velocity field
V	= volume of continuum
V_0	= one-dimensional initial velocity distribution
x_i	= space variables
y_f	= final transverse deformation
σ_{ij}	= generalized stress tensor
$\dot{\epsilon}_{ij}$	= generalized strain rate tensor
$\dot{\epsilon}_{ij}^*$	= assumed generalized strain rate tensor
$\dot{\epsilon}_{ij}^s$	= assumed time independent generalized strain rate tensor
λ	= Lagrangian multiplier
$\dot{\theta}$	= rotation rate
ρ	= mass per unit volume

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Introduction

ALTHOUGH there exist many problems of current interest involving the deformation of impacted and impulsively loaded continua, only a limited number, generally those with extremely simple geometry, have been solved exactly, and the more complex problems are highly intractable. Therefore, the availability of bounding techniques for this class of problems has real merit.

In a series of papers, [1–4] Martin has developed an upper bound technique for deformations of dynamically loaded structures. A major feature of his work is the capability of obtaining reasonable bounds with relatively simple mathematical operations. To this time, however, no comparable approach has been available to determine lower bounds on the displacements of dynamically loaded structures. The technique presented below thus complements Martin's upper bound theorem by providing a means to obtain lower bounds on the deformation of certain classes of dynamically loaded inelastic structures.

Lower Bound Theorem

Consider an inelastic body of volume V and surface S which at time $t < 0$ is assumed to be at rest. Let a velocity $\dot{u}_i^0$ be prescribed at all points in the continuum at time $t = 0$, and for time $t > 0$ it is assumed that the displacement rates $\dot{u}_i$ are zero on the portion of the surface S_u and tractions T_i are zero on the portion of the surface S_F. It is further assumed that the effect of body forces F_i is negligible in the process of deformation.

The response of the body at any time $t > 0$ may be characterized by time dependent velocity and acceleration fields $\dot{u}_i$, $\ddot{u}_i$ and time dependent stress and strain rate fields σ_{ij}, $\dot{\epsilon}_{ij}$. The stress field σ_{ij} and the strain rate field $\dot{\epsilon}_{ij}$ are associated by the rigid-perfectly plastic constitutive relation

$$\dot{\epsilon}_{ij} = \lambda\langle J(\sigma_{ij})\rangle \partial J/\partial\sigma_{ij} \tag{1}$$

in which J defines a convex yield surface and λ is a Lagrangian multiplier. Furthermore,

$$\begin{aligned}\langle J(\sigma_{ij})\rangle &= 0 \text{ for } J(\sigma_{ij}) < 0 \\ &= 1 \text{ for } J(\sigma_{ij}) = 0\end{aligned} \tag{2}$$

Stresses such that $J(\sigma_{ij}) > 0$ are not admitted under the assumption of perfect plasticity. The yield surface $J(\sigma_{ij}) = 0$ is assumed to be convex and to contain the origin of coordinates in stress space. If $\dot{\epsilon}_{ij}^*$ is a kinematically admissible strain rate field which is associated with a stress field σ_{ij}^* by means of Eq. (1), it follows from the assumption of convexity of the yield function that

$$(\sigma_{ij}^* - \sigma_{ij})\dot{\epsilon}_{ij}^* \geq 0 \tag{3}$$

where σ_{ij} is a stress field on or inside the yield surface $J(\sigma_{ij}^*) = 0$. This fundamental inequality has been proposed by Drucker[5] to establish a criteria of stability for perfectly plastic materials.

The strain rate field $\dot{\epsilon}_{ij}^*$ is a function of the space variables x_i and the time variable t and since Eq. (3) applies to each material element in the continuum, it can be integrated over the entire volume V and over some interval of time t_1 to t_2,

$$\int_V dV \int_{t_1}^{t_2} D(\dot{\epsilon}_{ij}^*)dt \geq \int_V dV \int_{t_1}^{t_2} \sigma_{ij}\dot{\epsilon}_{ij}^* dt \tag{4}$$

where $D(\dot{\epsilon}_{ij}^*)$ is defined as the rate of dissipation of internal energy which is unique for a chosen kinematically admissible strain rate field $\dot{\epsilon}_{ij}^*$ (Ref. 6) and is computed from

$$D(\dot{\epsilon}_{ij}^*) = \sigma_{ij}^*\dot{\epsilon}_{ij}^* \tag{5}$$

From the principle of virtual velocities,[7] if σ_{ij} is in internal equilibrium with the surface tractions T_i, body forces F_i and inertia forces $-\rho\ddot{u}_i$, where ρ is the mass per unit volume,

$$\int_V \sigma_{ij}\dot{\epsilon}_{ij}^* dV = \int_S T_i\dot{u}_i^* dS + \int_V F_i\dot{u}_i^* dV - \int_V \rho\ddot{u}_i\dot{u}_i^* dV \tag{6}$$

From Eq. (6), the right side of Eq. (4) may be written as

$$\int_V dV \int_{t_1}^{t_2} \sigma_{ij}\dot{\epsilon}_{ij}^* dt = \int_S dS \int_{t_1}^{t_2} T_i\dot{u}_i^* dt + \int_V dV \int_{t_1}^{t_2} F_i\dot{u}_i^* dt - \int_V dV \int_{t_1}^{t_2} \rho\ddot{u}_i\dot{u}_i^* dt \tag{7}$$

The type of loading to be imposed on the system requires that the first two terms on the right side of Eq. (7) be zero. Furthermore, since a rigid plastic medium is totally dissipative, the velocities in the inelastic continuum will vanish at some time t_f, which is denoted the response time of the system. Therefore, if the limits of integration of the time variable are taken from $t_1 = 0$ to $t_2 = t_f$, Eq. (7) can be written as

$$\int_V dV \int_0^{t_f} D(\dot{\epsilon}_{ij}^*)dt \geq -\int_V dV \int_0^{t_f} \rho\ddot{u}_i\dot{u}_i^* dt \tag{8}$$

The term on the right side of Eq. (8) is next integrated by parts with respect to the time variable as follows:

$$-\int_0^{t_f} \rho\ddot{u}_i\dot{u}_i^* dt = -\rho\dot{u}_i\dot{u}_i^*\Big|_0^{t_f} + \int_0^{t_f} \rho\ddot{u}_i^*\dot{u}_i dt \tag{9}$$

however,

$$\begin{aligned} \dot{u}_i &= 0 \text{ at } t = t_f \\ \dot{u}_i &= \dot{u}_i^0 \text{ at } t = 0 \end{aligned} \tag{10}$$

Hence,

$$-\int_0^{t_f} \rho\ddot{u}_i\dot{u}_i^* dt = \rho\dot{u}_i^0\dot{u}_i^*\Big|_{t=0} + \int_0^{t_f} \rho\ddot{u}_i^*\dot{u}_i dt \tag{11}$$

The second term on the right side of Eq. (11) is next integrated by parts to obtain,

$$-\int_0^{t_f} \rho\ddot{u}_i\dot{u}_i^* dt = \rho\dot{u}_i^0\dot{u}_i^*\Big|_{t=0} + \rho\ddot{u}_i^* u_i\Big|_0^{t_f} - \int_0^{t_f} \rho\dddot{u}_i^* u_i dt \tag{12}$$

However, at $t = 0$, $u_i = 0$, therefore,

$$-\int_0^{t_f} \rho\ddot{u}_i\dot{u}_i^* dt = \rho\dot{u}_i^0\dot{u}_i^*\Big|_{t=0} + \rho\ddot{u}_i^* u_i\Big|_{t_f} - \int_0^{t_f} \rho\dddot{u}_i^* u_i dt \tag{13}$$

Putting Eq. (13) into Eq. (8) and rearranging terms

$$-\int_V \rho\ddot{u}_i^* u_i\Big|_{t=t_f} dV \geq \int_V \rho\dot{u}_i^0\dot{u}_i^*\Big|_{t=0} dV - \int_V dV \int_0^{t_f} \rho\dddot{u}_i^* u_i dt - \int_V dV \int_0^{t_f} D(\dot{\epsilon}_{ij}^*)dt \tag{14}$$

The idea now is to select the kinematically admissible velocity field $\dot{u}_i^*$ in such a way as to produce a vanishing of the second term on the right side of Eq. (14). If $\dot{u}_i^*$ is assumed to be representable by a product of a time dependent function $\dot{T}^*(t)$ and a time independent function $U_i^*(x_i)$, i.e.,

$$\dot{u}_i^* = U_i^*(x_i)\dot{T}^*(t) = A^*Q_i^*(x_i)\dot{T}^*(t) \tag{15}$$

where A^* is an amplitude and Q_i^* a dimensionless function of the space variables x_i which represents the shape of the deformation mode, $\dddot{u}_i^*$ will vanish if $\dot{T}^*$ is chosen to be of the form,

$$\begin{aligned} \dot{T}^* &= (t_f^* - t)/t_f^*, \; 0 \leq t \leq t_f^* \\ \dot{T}^* &= 0, \; t > t_f^* \end{aligned} \tag{16}$$

where t_f^* is a constant yet to be determined. Hence,

$$\dot{u}_i^* = U_i^*[(t_f^* - t)/t_f^*] \tag{17}$$

Substituting Eq. (17) into Eq. (14)

$$\int_V \rho U_i^* u_i\Big|_{t=t_f} dV \geq t_f^* \times \left[\int_V \rho\dot{u}_i^0 U_i^* dV - \int_V dV \int_0^{t_f} D(\dot{\epsilon}_{ij}^*)dt\right] \tag{18}$$

Since the objective of this theorem is to obtain a bound on the maximum deformation that a structure undergoes at time $t = t_f$ when subjected to an impulsive loading, i.e.,

$$(u_i)_{\max}\big|_{t=tf} \geq \text{Lower Bound} \tag{19}$$

the inequality Eq. (18) is put in the form shown in Eq. (19) by recalling a result of integral calculus,[8] that if $f(x)$ and $g^*(x)$ are two continuous functions in the interval $a \leq x \leq b$ and if the maximum value of $f(x)$ in this range is denoted by M, then, provided $g^*(x)$ does not change sign in the interval $[a,b]$,

$$\int_a^b f(x)g^*(x)dx \leq M \int_a^b g^*(x)dx \tag{20}$$

where $M \geq f(x)$ for all x in $a \leq x \leq b$.

Denoting the three components of u_i and u_i^* as u, g, w and U^*, G^*, W^*, respectively, the left side of Eq. (18) can be written as

$$\int_V \rho U_i^* u_i\Big|_{t=t_f} dV = \int_V \rho U^* u\Big|_{t=t_f} dV + \int_V \rho G^* g\Big|_{t=t_f} dV + \int_V \rho W^* w\Big|_{t=t_f} dV \tag{21}$$

In order to obtain lower bounds on each of the three components of u_i, three separate choices of the components of the assumed kinematically admissible field U_i^* must be made. For example, if a bound for g is desired, U_i^* can be assumed to have components $U^* = W^* = 0$ and G^*. In this case Eq. (21) becomes

$$\int_V \rho U_i^* u_i\Big|_{t=t_f} dV = \int_V \rho G^* g\Big|_{t=t_f} dV \tag{22}$$

Applying the result of Eq. (20) to Eq. (22),

$$\int_V \rho G^* g\Big|_{t=t_f} dV \leq g_{\max}\big|_{t=tf} \int_V \rho G^* dV \tag{23}$$

Hence, from Eqs. (23) and (18) a lower bound for $g_{\max}$ is obtained from

$$g_{\max}|_{t=t_f} \geq \left(1/\int_V \rho G^* dV\right) t_f^* \times \left[\int_V \rho \dot{u}_i^0 U_i^* dV - \int_V dV \int_0^{t_f} D(\dot{\epsilon}_{ij}^*) dt\right] \quad (24)$$

Two other similar expressions are needed to bound the $u_{\max}$ and $w_{\max}$ of a body resulting from an impulsive loading. Therefore, the result of Eq. (24) can be generalized to

$$(u_i)_{\max}|_{t=t_f} \geq \left(1/\int_V \rho U_i^* dV\right) t_f^* \times \left[\int_V \rho \dot{u}_k^0 U_k^* dV - \int_V dV \int_0^{t_f} D(\dot{\epsilon}_{ij}^*) dt\right] \quad (25)$$

if the stated limitations on assumed field components are recognized.

The above result gives a lower bound on the maximum deflection $(u_i)_{\max}$ which a structure undergoes under the action of impulsive loading. In order to use the previous expression, information regarding the last term is needed, since at this stage both t_f and t_f^* are unknown. J. B. Martin[1] obtained a lower bound on the response time in the form

$$t_f \geq \left(\int_V \rho \dot{u}_i^0 \dot{u}_i^s dV\right) \Big/ \left(\int_V D(\dot{\epsilon}_{ij}^s) dV\right) \quad (26)$$

where $\dot{u}_i^s$ denotes a time independent kinematically admissible velocity field and $\dot{\epsilon}_{ij}^s$ the associated time independent strain rate field. If t_f^* is chosen to be equal to the lower bound of t_f shown in Eq. (26), i.e.,

$$t_f^* = \left(\int_V \rho \dot{u}_i^0 \dot{u}_i^s dV\right) \Big/ \left(\int_V D(\dot{\epsilon}_{ij}^s) dV\right) \quad (27)$$

then

$$t_f \geq t_f^* \quad (28)$$

The result shown in Eq. (28) has been demonstrated by the technique proposed by Martin and Symonds.[9]

Hence, since $\dot{u}_i^*$ is zero for $t \geq t_f^*$ the last term of Eq. (25) can be evaluated as

$$\int_V dV \int_0^{t_f} D(\dot{\epsilon}_{ij}^*) dt = \int_V dV \int_0^{t_f^*} D(\dot{\epsilon}_{ij}^*) dt + \int_V dV \int_{t_f^*}^{t_f} D(\dot{\epsilon}_{ij}^*) dt = \int_V dV \int_0^{t_f^*} D(\dot{\epsilon}_{ij}^*) dt \quad (29)$$

Therefore, the lower bound theorem becomes

$$(u_i)_{\max}|_{t=t_f} \geq \left(1/\int_V \rho U_i^* dV\right) t_f^* \times \left[\int_V \rho \dot{u}_k^0 U_k^* dV - \int_V dV \int_0^{t_f^*} D(\dot{\epsilon}_{ij}^*) dt\right] \quad (30)$$

Equation (30) thus represents a usable lower bound theorem for the maximum deformation that a multi-dimensional system undergoes as a result of an impulsive loading.

Application to Example Problems

The theorem will now be illustrated by applying it to several problems, involving impulsive loads, whose exact solutions have been obtained and the results will be compared both with the exact solutions and with upper bound values obtained by Martin.

Example Problem 1

Consider a simply supported beam of span 2L and mass m per unit length. The beam will be assumed to be rigid-perfectly plastic with a limiting or yield bending moment M_0. Let this beam be subjected to a uniform velocity V_0 at time = 0. Symonds[10] found the maximum transverse deflection

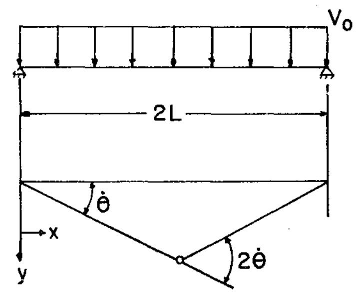

Fig. 1 Assumed deformation mode Y*.

to be

$$y_f = \tfrac{1}{3}\,(mL^2V_0^2/M_0) \quad (31)$$

In order to compute a lower bound on y_f, a kinematically admissible velocity mode is assumed and the response time t_f^* associated with this mode is obtained. One such possibility is to assume that a hinge is formed at the center of the beam and that the halves of the beam rotate as rigid bodies with rotation rate $\dot{\theta}$. Hence, at any distance x from the end of each of the halves of the beam the velocity is given by

$$Y^* = x\dot{\theta} \qquad 0 \leq x \leq L \quad (32)$$

Figure 1 shows the beam undergoing a deformation described by Eq. (32) where the total rotation rate at the center of the beam is $2\dot{\theta}$.

From Eq. (27), the response time associated with the assumed deformation mode Y^* is given by

$$t_f^* = \left(2\int_0^L mV_0(x\dot{\theta})dx\right) \Big/ M_0(2\dot{\theta}) = \frac{1}{2}\left(\frac{mL^2V_0}{M_0}\right) \quad (33)$$

A lower bound on the maximum deformation in the transverse direction is obtained from Eq. (30)

$$y_f \geq 1\Big/\left(2\int_0^L mx\dot{\theta}dx\right)\left\{\frac{1}{2}\left(\frac{mL^2V_0}{M_0}\right) \times \left[2\int_0^L mV_0x\dot{\theta}dx - \frac{1}{2}\left(\frac{\frac{1}{2}mL^2V_0}{M_0}\right)(2M_0\dot{\theta})\right]\right\} \quad (34)$$

$$y_f \geq \tfrac{1}{4}(mL^2V_0^2/M_0) \quad (35)$$

Martin[1] computed an upper bound of y_f to be,

$$y_f \leq \tfrac{1}{2}(mL^2V_0^2/M_0) \quad (36)$$

Therefore,

$$\tfrac{1}{4}(mL^2V_0^2/M_0) \leq y_f \leq \tfrac{1}{2}(mL^2V_0^2/M_0) \quad (37)$$

where

$$y_f = \tfrac{1}{3}(mL^2V_0^2/M_0)$$

Example Problem 2

Consider a simply supported rigid-perfectly plastic circular plate of radius R which at time $t = 0$ is assumed to be subjected to a uniform velocity V_0. Let the circumferential and radial plastic moment be M_0 and adopt the Tresca's yield condition to govern the behavior of the plate.

In order to obtain a lower bound on the maximum deformation at time t_f, assume a kinematically admissible mode shape Y^* of the form,

$$Y^* = (R - r)/R \quad (38)$$

The response time of the system corresponding to the assumed mode is given by

$$t_f^* = \left(\int_0^R m_A V_0 Y^* 2\pi r dr\right) \Big/ \int_0^R M_0(\dot{k}_r^* + \dot{k}_\theta^*) \times 2\pi r dr = \tfrac{1}{6}(m_A R^2 V_0/M_0) \quad (39)$$

where the radial and circumferential curvature rates $\dot{k}_r^*$ and $\dot{k}_\theta^*$ corresponding to the assumed mode shape Y^* are computed from

$$\dot{k}_r^* = -(d^2Y^*/dr^2),\ \dot{k}_\theta^* = -(1/r)(dY^*/dr) \quad (40)$$

The lower bound on the maximum transverse deformation at time $t = t_f$ can now be computed from Eq. (30)

$$y_f \geq (1)/(\tfrac{1}{3}\pi m_A R^2 V_0)\left(\tfrac{1}{6}(m_A R^2 V_0/M_0)\{\tfrac{1}{3}(\pi m_A R^2 V_0/M_0) - [\tfrac{1}{12}(m_A R^2 V_0/M_0)]\cdot(2\pi M_0)\}\right) \quad (41)$$

$$y_f \geq \tfrac{1}{12}(m_A R^2 V_0^2/M_0) \quad (42)$$

The upper bound on y_f has been obtained by Martin[1] as

$$y_f \leq \tfrac{1}{4}(m_A R^2 V_0^2/M_0) \quad (43)$$

Therefore,

$$\tfrac{1}{12}(m_A R^2 V_0^2/M_0) \leq y_f \leq \tfrac{1}{4}(m_A R^2 V_0^2/M_0) \quad (44)$$

The exact solution to this problem was given by Hopkins and Prager[11] as

$$y_f = \tfrac{1}{8}(m_A R^2 V_0^2/M_0)$$

Example Problem 3

A point mass M is attached to the surface of a rigid-perfectly plastic beam of length L and negligible mass m per unit length compared to M. At time $t = 0$, the mass M is given an initial velocity V_0 in the vertical direction, while the remainder of the structure remains stationary. As in the previous examples, a kinematically admissible mode shape Q_i^* is assumed and the response time t_f^* of the system corresponding to this assumed mode is computed. One such possibility is to assume a hinge is formed at the base of the beam and that the entire beam rotates about the hinge with a rotation rate $\dot{\theta}$. Therefore, at any distance x from the base of the beam the assumed mode shape is given by

$$Y^* = x\dot{\theta} \qquad 0 \leq x \leq L \quad (45)$$

From Eq. (27) the response time t_f^* corresponding to the assumed mode shape Y^* is given by

$$t_f^* = \left(\int_0^L mV_0 x\dot{\theta}dx + \frac{MV_0L\dot{\theta}}{M_0\dot{\theta}}\right) = \left(\frac{MLV_0}{M_0}\right) \quad (46)$$

The lower bound on the displacement at the free end can now be calculated from Eq. (30)

$$y_f \geq 1/ML\dot{\theta}\{(MLV_0/M_0)[MV_0L\dot{\theta} - \tfrac{1}{2}(MLV_0/M_0)M_0\dot{\theta}]\} \quad (47)$$

$$y_f \geq \tfrac{1}{2}(MLV_0^2/M_0) \quad (48)$$

Martin[1] determined y_f to be smaller than or equal to $\frac{1}{2}(MLV_0^2/M_0)$. Hence,

$$\tfrac{1}{2}(MLV_0^2/M_0) \leq y_f \leq \tfrac{1}{2}(MLV_0^2/M_0) \quad (49)$$

The exact solution to this problem was obtained by Parkes[12] to be

$$y_f = \tfrac{1}{2}(MLV_0^2/M_0) \quad (50)$$

Summary

Through the use of Drucker's stability postulate for time independent inelastic materials and the virtual work equation a technique has been developed which bounds from below the deformation of dynamically loaded rigid-plastic bodies, thus complementing the upper bound theorem derived by Martin. As with Martin's work, this technique is limited to small deformations. The lower bound theorem has been applied to several problems involving rigid-perfectly plastic bodies subjected to impulsive loads and the results compared to the exact solutions and the values obtained from Martin's upper bound theorem. In all the problems considered, the values obtained from the lower bound theorem were at least as close to the exact result as the upper bound results. In any event, for either theorem, these values were readily obtained from relatively simple mathematical operations, and the simplicity with which bounds are calculated is an attractive feature of the technique presented.

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